On the Logical Geometry of the Tetrahedron

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1 Introduction

The central claim we[1] would like to defend in the present paper is that the algebraic analysis of opposition and implication relations between propositions, operators, or quantifiers (Jaspers, 2005; Smessaert, 2009; Smessaert & Demey, 2014) not only carries over to the realms of colour, numbers or natural language concepts (Jaspers, 2012; Seuren & Jaspers, 2014), but also to that of (3D) geometrical shapes. To corroborate this claim we will present a detailed logical-algebraic and geometrical analysis of the simplest of all polyhedra, viz. the TETRAHEDRON. We start off by introducing the basic geometric properties of the tetrahedron (§ 2). Then we introduce two algebras – one for the tetrahedron's (zero-dimensional) VERTEX constellations (§ 3) and one for its (two-dimensional) FACE constellations (§ 4) – and go into two possible isomorphisms between the two algebras (§ 5). Finally, we turn to the more complex algebra for the tetrahedron's (one-dimensional) EDGE constellations (§ 6) and briefly consider some partial mappings between the vertex/face algebras on the one hand and the edge algebra on the other (§ 7).

2 Basic properties of the tetrahedron

Platonic solid	vertices V	edges E	faces F	Euler V - E + F = 2
TETRAHEDRON	4	6	4 (triangular)	4 - 6 + 4 = 2
HEXAHEDRON	8	12	6 (square)	8 - 12 + 6 = 2
OCTAHEDRON	6	12	8 (triangular)	6 - 12 + 8 = 2

Table 1: The Euler formula for the tetrahedron, hexahedron and octahedron.

The TETRAHEDRON – informally speaking a 'pyramid with a triangular base plane' – is the smallest 3D shape possible. It has 4 vertices (V), 4 equilateral triangular faces

[1] It seemed like an appropriate move to make Dany the co-author of a paper in his own Festschrift, even though he has largely been unaware of the paper's existence or its writing process. The ideas that are explored here are the result of many a lively discussion we had together on our shared interest in algebra and geometry. Let it be a token of my gratitude for our past and continued collaboration and friendship (HS).

(F), and 6 edges (E), and satisfies the Euler Formula V - E + F = 2. It belongs to the group of PLATONIC POLYHEDRA, which includes the hexahedron (cube), the octahedron, the dodecahedron and the icosahedron.[2] The three smallest Platonic solids are described in Table 1. The hexahedron and octahedron (and similarly the dodecahedron and icosahedron) are one another's DUALS: connecting the mid points of the 6 faces of a hexahedron yields an octahedron, and conversely, connecting the mid points of the 8 faces of an octahedron yields a hexahedron. In other words, the duality is based on 'exchanging vertices and faces'. The tetrahedron is called SELF-DUAL: connecting the mid points of the 4 faces of a tetrahedron again yields a tetrahedron[3]

Although the tetrahedron is defined as a 3D solid, it is standardly characterized — as was the case above — on the lower dimensions in terms of three sets, namely the set of its vertices V, the set of its edges E and the set of its faces F. In the algebraic approach adopted in this paper, edges are 'created' on the basis of two vertices by means of the two-place edge-operator $\circ$, and faces are 'created' on the basis of three vertices by means of the (discontinuous) three-place face-operator $\bullet \ldots \bullet$. Both operators will be used in their infix-notation, i.e. $x \circ y$ is the edge connecting vertices x and y, and $x \bullet y \bullet z$ is the triangular face connecting the vertices x, y and z:

0-dimensionality	vertices	V	:=	$\{a, b, c, d\}$
1-dimensionality	edges	E	:=	$\{a \circ b, a \circ c, a \circ d, b \circ c, b \circ d, c \circ d\}$
2-dimensionality	faces	F	:=	$\{a \bullet b \bullet c, a \bullet b \bullet d, a \bullet c \bullet d, b \bullet c \bullet d\}$

3 The vertex algebra VA (zero-dimensionality)

On the basis of the set of vertices V we can generate the set VC of all vertex-constellations (vc) as the power set of V, i.e. VC := $\wp$(V). The VC set then serves as the first element of the six-tuple, making up the VERTEX ALGEBRA VA, a classical Boolean Algebra, with the binary operators of intersection and union ($\cap$ and $\cup$), the unary complement operator ($\sim$), and the bottom and top elements ($\varnothing$ and V):

$$\text{VA} := < \text{VC}, \cap, \cup, \sim, \varnothing, \text{V} >$$

This algebra VA can be visually represented by means of a lattice (Davey & Priestley, 2002) or Hasse diagram (Demey & Smessaert, 2014), as is shown informally — i.e. without visualising the partial ordering relations among the elements[4] — in Figures 1 and 2. The $2^4 = 16$ vc's in VC can be subdivided into 5 LEVELS — levels zero ($L0$) through four ($L4$) – according to the number of their constituent vertices:[5]

The algebraic analysis of the Aristotelian relations — and their 3D visualisation by means of the RHOMBIC DODECAHEDRON (RDH) — straightforwardly carries over to the

[2]See standard reference works such as Cromwell (1997) and Coxeter (1973) for more details.

[3]In other words, with self-dual polyhedra the number of vertices equals the number of faces.

[4]For convenience's sake, we replace the curly brackets and comma's for the subsets of a set by the plus symbol, i.e. $a + \ldots + b := \{a, \ldots, b\}$, and hence $a + b + c + d = \{a, b, c, d\} = V$, i.e. the top element of the VA algebra.

[5]This constellation/distribution is well-known as that of the binomial coefficients, as arranged in *Pascal's triangle* (Edwards, 2002)

$L4$	1			$a+b+c+d$			
$L3$	4		$a+b+c$	$a+b+d$	$a+c+d$	$b+c+d$	
$L2$	6	$a+b$	$a+c$	$a+d$	$b+c$	$b+d$	$c+d$
$L1$	4		a	b	c	d	
$L0$	1			$\varnothing$			

Figure 1: The Hasse diagram of the Vertex Algebra VA for the tetrahedron.

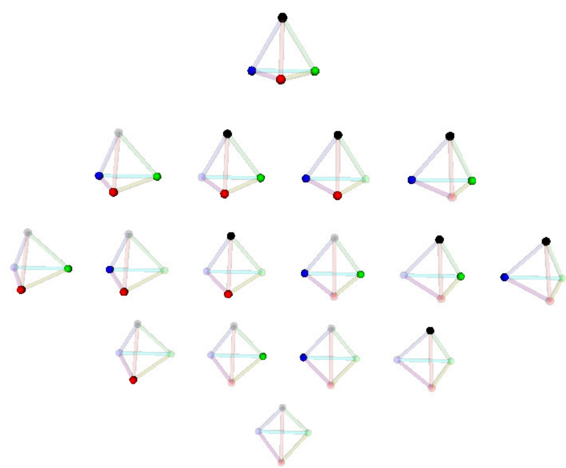

Figure 2: The 3D Hasse diagram of the Vertex Algebra VA for the tetrahedron.

domain of vertex constellations VC.[6] Two vc's can be said to stand in the Aristotelian relations of CONTRADICTION, (SUB)CONTRARIETY, or SUBALTERNATION. For any pair of vc's φ and $\psi \in VC$ it holds that they are:

$$\begin{array}{rcrcccccccccccccc}
2^0 = & & 1 & = & & & & & & & 1 & & & & & & \\
2^1 = & & 2 & = & & & & & & 1 & & 1 & & & & & \\
2^2 = & & 4 & = & & & & & 1 & & 2 & & 1 & & & & \\
2^3 = & & 8 & = & & & & 1 & & 3 & & 3 & & 1 & & & \\
2^4 = & & 16 & = & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
2^5 = & & 32 & = & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
2^6 = & & 64 & = & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

[6] In particular, the vertex algebra VA is isomorphic to the Boolean algebra B_4 (Smessaert & Demey, 2016; Demey & Smessaert, 2017b), and could therefore receive an analysis in terms of *bitstrings* of length 4 (Smessaert & Demey, 2017; Demey & Smessaert, 2017a). For more details, see `www.logicalgeometry.org`.

contradictory	iff	$\varphi \cap \psi = \emptyset$	and $\varphi \cup \psi = V$
contrary	iff	$\varphi \cap \psi = \emptyset$	and $\varphi \cup \psi \neq V$
subcontrary	iff	$\varphi \cap \psi \neq \emptyset$	and $\varphi \cup \psi = V$
in subalternation	iff	$\varphi \cap \psi = \varphi$	and $\varphi \cup \psi \neq \varphi$

Using these definitions, the $L2$ vc $a+b$, for instance, is contradictory to $L2$ $c+d$, contrary to $L1$ c and d, subcontrary to $L3$ $a+c+d$ and $b+c+d$ and in subalternation with $L1$ a and b. Note that the relation of contradiction is represented by means of CENTRAL SYMMETRY, both in the 'symbolic' version in Figure 1 and in the 3D graphical version of Figure 2.[7] In other words, any vc has its contradictory located at the mirror image position with respect to an (imaginary) point of symmetry in the middle of the visual representation.

The purpose of the present paper, however, is not so much to study the *internal* structure of the vertex algebra VA as such, but rather the *external* relations with two different algebras, namely that of the faces (two-dimensionality) of the tetrahedron and that of its edges (one-dimensionality).

4 The face algebra FA (two-dimensionality)

Recall that F is the set consisting of the 4 faces of the tetrahedron $\{a \bullet b \bullet c, a \bullet b \bullet d, a \bullet c \bullet d, b \bullet c \bullet d\}$. On the basis of F we can generate the power set $\wp(F)$ = FC of all face-constellations (fc), which then serves as the first element of the FACE ALGEBRA FA, again a classical Boolean Algebra:

$$\text{FA} := < \text{FC}, \cap, \cup, \sim, \emptyset, \text{F} >$$

As was the case for the vertex algebra VA in the previous section, the face algebra FA can be visually represented by means of the Hasse diagrams in Figures 3 and 4. The $2^4 = 16$ fc's in FC can be subdivided into 5 LEVELS — $L0$ through $L4$ – according to the number of their constituent faces:

$L4$	1	$a \bullet b \bullet c + a \bullet b \bullet d + a \bullet c \bullet d + b \bullet c \bullet d$
$L3$	4	$a \bullet b \bullet c + a \bullet b \bullet d + a \bullet c \bullet d$ $\ldots$ $\ldots$ $a \bullet b \bullet d + a \bullet c \bullet d + b \bullet c \bullet d$
$L2$	6	$a \bullet b \bullet c + a \bullet b \bullet d$ $\ldots$ $\ldots$ $\ldots$ $\ldots$ $a \bullet c \bullet d + b \bullet c \bullet d$
$L1$	4	$a \bullet b \bullet c$ $a \bullet b \bullet d$ $a \bullet c \bullet d$ $b \bullet c \bullet d$
$L0$	1	$\emptyset$

Figure 3: The Hasse diagram of the Face Algebra FA for the tetrahedron.

[7] The interactive 3D virtual reality versions (in X3D format) of Figures 2, 4 and 6 are available from `www.logicalgeometry.org/3D-diagrams.htm`.

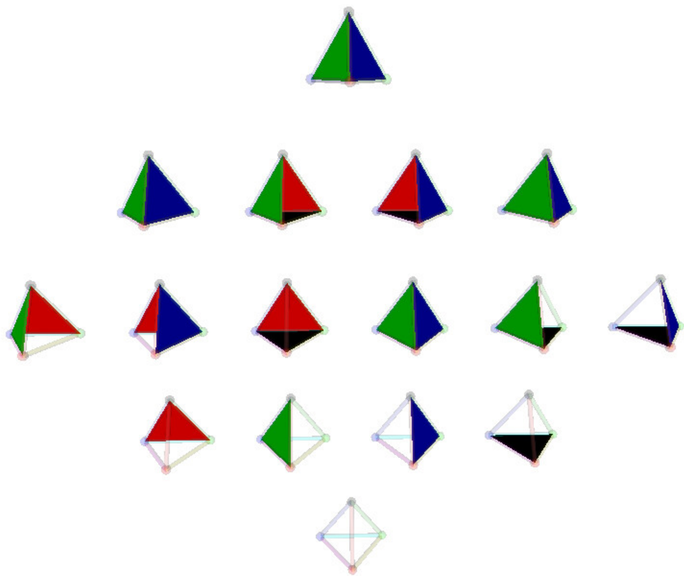

Figure 4: The 3D Hasse diagram of the Face Algebra FA for the tetrahedron.

The algebraic definitions of the Aristotelian relations — as given for the vertex constellations in the previous section — straightforwardly carry over to the face constellations of FC. The $L2$ fc $a \bullet b \bullet c + a \bullet b \bullet d$, for instance, is contradictory to $L2$ $a{\bullet}c{\bullet}d{+}b{\bullet}c{\bullet}d$, contrary to $L1$ $a{\bullet}c{\bullet}d$ and subcontrary to $L3$ $a{\bullet}b{\bullet}d{+}a{\bullet}c{\bullet}d{+}b{\bullet}c{\bullet}d$. Notice that — both in Figures 3 and 4 — any fc has its contradictory located at the mirror image position with respect to the point of central symmetry in the visual representation. As pointed out before, however, the *internal* structure of the individual algebras is not our main concern here. Instead, we want to focus on the *external* relations between the algebras. In the next section, we go into the systematic correlations between the two algebras introduced so far, namely the vertex algebra VA for zero-dimensionality and the face algebra FA for two-dimensionality.

5 Isomorphisms between VA and FA

Due to the self-duality of the tetrahedron, the number of vertices is identical to the number of faces. As a consequence, the two algebras for the vertex and face constellations — VA and FA — have the same degree of complexity: they are both isomorphic to the Boolean algebra B_4 (with bitstrings of length 4). Applying transitivity, we can now consider the isomorphism between VA and FA.[8]

[8] Although this notion of isomorphism can easily be made formally precise, we will refrain from doing so in the present context, focussing on the underlying intuitions instead. We will informally represent the isomorphism relation with the similarity symbol $\sim$.

As a matter of fact, at least two 'natural' ways of establishing this isomorphism suggest themselves, a 'parallel but negative' way and a 'diagonal but positive' way. With the PARALLEL isomorphism, the notion of level is maintained in that the 4 $L1$ elements of VA are mapped onto the 4 $L1$ elements of FA, and similarly for the respective $L2$ and $L3$ elements. With the DIAGONAL isomorphism, by contrast, the $L1$ elements of VA are mapped onto the $L3$ of FA, $L3$ of VA onto $L1$ of FA, and $L2$ of VA onto $L2$ of FA. Furthermore, the parallel isomorphism can be called a NEGATIVE mapping, since it establishes a relationship of *complementarity* or *opposition*, whereas the diagonal isomorphism can be called a POSITIVE mapping, since it establishes a relationship of *connection/constitution* or *implication*.[9] Let us first consider the *parallel* — $L1 \sim L1$, $L2 \sim L2$ and $L3 \sim L3$ — but *negative* — oppositional — isomorphism between the VA and the FA:

- mapping the 4 $L1$ elements of VA onto the 4 $L1$ elements of FA, each vertex is associated with the opposite face that consists of the remaining three vertices:

$$\begin{array}{lclclcl} a & \sim & b \bullet c \bullet d & \qquad & b & \sim & a \bullet b \bullet d \\ c & \sim & a \bullet b \bullet d & & d & \sim & a \bullet b \bullet c \end{array}$$

- mapping the 4 $L3$ elements of VA onto the 4 $L3$ elements of FA, each trio of vertices is associated with the trio of faces which excludes the plane defined by the three vertices:

$$\begin{array}{lcl} a+b+c & \sim & a \bullet b \bullet d + b \bullet c \bullet d + a \bullet c \bullet d \\ a+b+d & \sim & a \bullet b \bullet c + b \bullet c \bullet d + a \bullet c \bullet d \\ a+c+d & \sim & a \bullet b \bullet c + b \bullet c \bullet d + a \bullet b \bullet d \\ b+c+d & \sim & a \bullet b \bullet c + a \bullet c \bullet d + a \bullet b \bullet d \end{array}$$

- mapping the 6 $L2$ elements of VA onto the 6 $L2$ elements of FA, each pair of vertices is associated with the pair of faces which excludes the edge defined by the two vertices:

$$\begin{array}{lclclcl} a+b & \sim & b \bullet c \bullet d + a \bullet c \bullet d & \qquad & b+c & \sim & a \bullet c \bullet d + a \bullet b \bullet d \\ a+c & \sim & b \bullet c \bullet d + a \bullet b \bullet d & & b+d & \sim & a \bullet b \bullet c + a \bullet c \bullet d \\ a+d & \sim & a \bullet b \bullet c + b \bullet c \bullet d & & c+d & \sim & a \bullet b \bullet c + a \bullet b \bullet d \end{array}$$

We can now compare the above 'parallel but negative' isomorphism to the *diagonal* — $L1 \sim L3$, $L2 \sim L2$ and $L3 \sim L1$ — but *positive* — implicational — isomorphism between the VA and the FA below:

- mapping the $L1$ elements of VA onto the $L3$ elements of FA, each vertex is associated with the trio of faces that it constitutes the 'apex' of:

$$\begin{array}{lcl} a & \sim & a \bullet b \bullet c + a \bullet c \bullet d + a \bullet b \bullet d \\ b & \sim & a \bullet b \bullet c + b \bullet c \bullet d + a \bullet b \bullet d \\ c & \sim & a \bullet b \bullet c + b \bullet c \bullet d + a \bullet c \bullet d \\ d & \sim & a \bullet b \bullet d + b \bullet c \bullet d + a \bullet c \bullet d \end{array}$$

[9] A possibly interesting parallellism suggests itself here with the distinction drawn in Smessaert & Demey (2014) between the *opposition* and *implication geometries*.

- mapping the $L3$ elements of VA onto the $L1$ elements of FA, each trio of vertices is associated with the face it constitutes:

$$\begin{array}{lcl@{\qquad}lcl} a+b+c & \sim & a\bullet b\bullet c & a+b+d & \sim & a\bullet b\bullet d \\ a+c+d & \sim & a\bullet c\bullet d & b+c+d & \sim & b\bullet c\bullet d \end{array}$$

- mapping the $L2$ elements of VA onto the $L2$ elements of FA, each pair of vertices is associated with the pair of faces which are joined at the edge constituted by the two vertices:

$$\begin{array}{lcl@{\qquad}lcl} a+b & \sim & a\bullet b\bullet c+a\bullet b\bullet d & b+c & \sim & a\bullet b\bullet c+b\bullet c\bullet d \\ a+c & \sim & a\bullet b\bullet c+a\bullet c\bullet d & c+d & \sim & b\bullet c\bullet d+a\bullet c\bullet d \\ a+d & \sim & a\bullet c\bullet d+a\bullet b\bullet d & b+d & \sim & b\bullet c\bullet d+a\bullet b\bullet d \end{array}$$

Both the 'parallel but negative' isomorphism and the 'diagonal but positive' isomorphism yielded perfect one-to-one correspondences between the vertex constellations of VA and the face constellations of FA. In the next section, we introduce the more complex edge algebra EA, which will only enter into *partial* correspondence relations with either VA or FA.

6 The edge algebra EA (one-dimensionality)

Recall that E is the set consisting of the 6 edges of the tetrahedron $\{a\circ b, a\circ c, a\circ d, b\circ c, b\circ d, c\circ d\}$. On the basis of E we can generate the power set $\wp(E)$ = EC of all edge-constellations (ec), which then serves as the first element of the EDGE ALGEBRA EA, again a classical Boolean Algebra:

$$\text{EA} := <\text{EC}, \cap, \cup, \sim, \varnothing, \text{E}>$$

In contrast to the VC and FC algebras — which contained $2^4 = 16$ vc's and fc's in Sections 3 and 4 respectively — the number of ec's is considerably larger, i.e. $2^6 = 64$. These 64 ec's can be subdivided into 7 levels ($L0$ through $L6$), according to the number of their constituent edges.[10] The edge algebra EA is represented visually — but schematically, i.e. partially — by means of the Hasse diagrams in Figures 5 and 6:

Strictly speaking, the algebraic definitions of the Aristotelian relations — as given for the vertex constellations in Section 3 — straightforwardly carry over to the edge constellations of EA. However, EA is no longer isomorphic to the Boolean algebra B_4 (with bitstrings of length 4) but rather to B_6, i.e. with bitstrings of length 6. The latter structure is clearly far less well-studied within Logical Geometry than the rhombic dodecahedron for B_4, so we will not go into any further Aristotelian details here.

Nevertheless, one important observation needs to be made at this point: the step from the 16 vertex/face constellations to the 64 edge constellations has 'broken the uniformity' within the levels. In VA and FA, all elements that occur on one level are of the same 'type', i.e. identical up to permutation or rotation. This is obvious for the

[10] In other words, the complexity increases considerably when we move from the fifth row of *Pascal's triangle* down to the seventh row (cf. footnote 5).

$L6$	1	$a \circ b + a \circ c + a \circ d + b \circ c + b \circ d + c \circ d$
$L5$	6	$a \circ b + \ldots + \ldots + \ldots + b \circ d \quad \ldots \quad a \circ c + \ldots + \ldots + \ldots + c \circ d$
$L4$	15	$a \circ b + a \circ c + a \circ d + b \circ c \quad \ldots \quad \ldots \quad a \circ d + b \circ c + b \circ d + c \circ d$
$L3$	20	$a \circ b + a \circ c + a \circ d \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad b \circ c + b \circ d + c \circ d$
$L2$	15	$a \circ b + a \circ c \quad a \circ b + a \circ d \quad \ldots \quad b \circ c + c \circ d \quad b \circ d + c \circ d$
$L1$	6	$a \circ b \quad a \circ c \quad a \circ d \quad b \circ c \quad b \circ d \quad c \circ d$
$L0$	1	$\emptyset$

Figure 5: The Hasse diagram of the Edge Algebra EA for the tetrahedron.

4 single elements or the 4 trio's that occur on the respective $L1$ and $L3$ of both VA and FA, but even the 6 pairs of vertices or the 6 pairs of faces on the middle $L2$ are clearly uniform. With the considerably bigger numbers of edge constellations on the middle three levels of EA, however — namely $L2$, $L3$ and $L4$ in Figures 5 and 6 — this uniformity is lost. The 15 ec's that occur on both $L2$ and $L4$ turn out to fall apart into a small subgroup of 3 ec's and the remaining group of 12 ec's:

$L2$	3	$a \circ b + c \circ d \quad a \circ c + b \circ d \quad a \circ d + b \circ c$
	12	$a \circ b + a \circ c \quad a \circ b + a \circ d \quad \ldots \quad \ldots \quad b \circ c + c \circ d \quad b \circ d + c \circ d$
$L4$	3	$a \circ c + a \circ d + b \circ c + b \circ d \quad a \circ b + a \circ d + b \circ c + c \circ d$
		$a \circ b + a \circ c + b \circ d + c \circ d$
	12	$a \circ d + b \circ c + b \circ d + c \circ d \quad \ldots \quad \ldots \quad a \circ b + a \circ c + a \circ d + b \circ c$

The three ec's that stand apart on $L2$ are precisely the 'unconnected' pairs of edges that do not share any vertex, whereas the remaining 12 pairs are the 'V-shaped' ec's, i.e. pairs of edges that have one vertex in common. The three ec's that stand apart on $L4$, by contrast, are the 'double V-shaped' four edge ec's yielding a 'closed circuit', whereas the remaining 12 ec's consist of a triangular face with one extra edge 'sticking out'.[11] The lack of uniformity even increases when we move to the $L3$ elements of the EA, since the 20 $L3$ ec's break up into two small subsets of 4 members and a bigger subset containing the remaining 12 ec's:

$L3$	4	$a \circ b + a \circ c + b \circ c$	$a \circ b + a \circ d + b \circ d$
		$a \circ c + a \circ d + c \circ d$	$b \circ c + b \circ d + c \circ d$
	4	$a \circ d + b \circ d + c \circ d$	$a \circ c + b \circ c + c \circ d$
		$a \circ b + b \circ c + b \circ d$	$a \circ b + a \circ c + a \circ d$
	12	$a \circ b + b \circ c + c \circ d \quad \ldots \quad \ldots$	$a \circ c + a \circ d + b \circ d$

The first 4 $L3$ ec's consist of three edges constituting a face of the tetrahedron, whereas the second group of 4 $L3$ ec's consist of three edges sharing one vertex and

[11] Not surprisingly, the 3 special $L2$ ec's have a contradictory ec among the 3 special $L4$ ones.

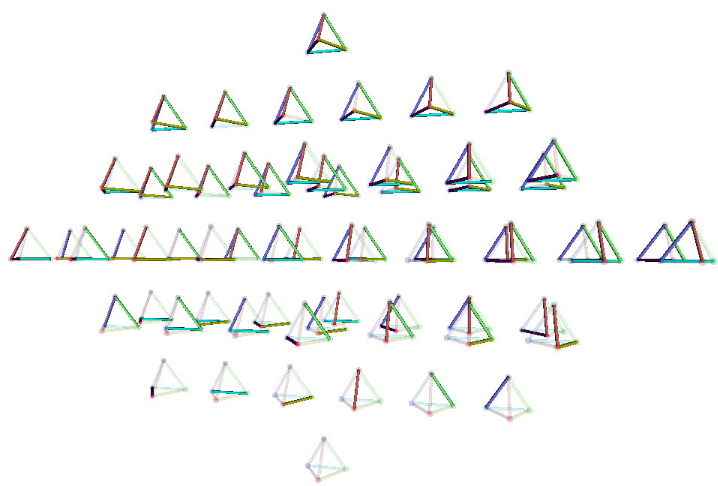

Figure 6: The 3D Hasse diagram of the Edge Algebra EA for the tetrahedron.

hence constituting a 'tripod'.[12] The remaining 12 ec's of $L3$ yield 'S or Z-shaped' constellations with three consecutively connected edges not constituting a closed circuit.[13] The lack of uniformity — in particular that on $L3$ of the EA — will play a crucial role in the next section when describing the partial mapping relations between the EA on the one hand and VA or FA on the other.

7 Mappings between VA/FA and EA

Let us, in a final step, consider possible mappings between the complex EA discussed in the previous section and the two more basic algebras VA and FA. Given the huge numerical discrepancy between the 16 vc's or fc's and the 64 ec's, these mappings will — of necessity — be *partial*, the idea being that all 16 of the former get mapped onto one of the latter, but not vice versa. The distinction drawn in Section 5 between 'negative' mappings — relations of complementarity or opposition — and 'positive' mappings — relations of constitution or implication — can be used here as well. Furthermore, this contrast shows up both in the VA to EA mappings and in the FA to EA mappings. By way of illustration, we will consider two possible combinations. First we look at a partial negative mapping from VA to EA:

- mapping the 4 $L1$ elements of VA onto 4 of the $L3$ elements of EA, each vertex is associated with the three edges that connect the remaining three vertices into a face of the tetrahedron:

[12] Again not surprisingly, these two small subsets of 4 $L3$ ec's yield 4 pairs of contradictory ec's.

[13] This raises the intriguing question as to the possible connection between the *graphical* idea of unconnectedness and the *logical* notion of unconnectedness introduced in Smessaert & Demey (2014).

$$
\begin{array}{lcl}
a & \sim & b\circ c+c\circ d+d\circ b \\
c & \sim & a\circ b+b\circ d+d\circ a
\end{array}
\qquad
\begin{array}{lcl}
b & \sim & a\circ c+c\circ d+d\circ a \\
d & \sim & a\circ b+b\circ c+c\circ a
\end{array}
$$

- mapping the 4 $L3$ elements of VA onto 4 of the $L3$ elements of EA, each trio of vertices is associated with the trio of edges which do not belong to the face constituted by the three vertices:

$$
\begin{array}{lcl}
a+b+c & \sim & a\circ d+b\circ d+c\circ d \\
a+c+d & \sim & a\circ b+c\circ b+d\circ b
\end{array}
\qquad
\begin{array}{lcl}
a+b+d & \sim & a\circ c+b\circ c+d\circ c \\
b+c+d & \sim & b\circ a+c\circ a+d\circ a
\end{array}
$$

- mapping the 6 $L2$ elements of VA onto the 6 $L1$ elements of EA, each pair of vertices is associated with the edge constituted by the two excluded vertices:

$$
\begin{array}{lcl}
a+b & \sim & c\circ d \\
b+d & \sim & a\circ c
\end{array}
\qquad
\begin{array}{lcl}
b+c & \sim & a\circ d \\
a+d & \sim & b\circ c
\end{array}
\qquad
\begin{array}{lcl}
a+c & \sim & b\circ d \\
c+d & \sim & a\circ b
\end{array}
$$

Notice that the two special subsets of 4 $L3$ ec's characterised above show up when mapping the $L1$ and $L3$ elements of VA. Furthermore, there is a perfect one-to-one mapping between the 6 vc's of the most complex $L2$ of VA and the 6 ec's on the simplest $L1$ of EA. Secondly, we consider the partial positive mapping from FA to EA:

- mapping the 4 $L1$ elements of FA onto 4 of the $L3$ elements of EA, each face is associated with the trio of edges whose vertices constitute the face:

$$
\begin{array}{lcl}
a\bullet b\bullet c & \sim & a\circ b+b\circ c+c\circ a \\
a\bullet c\bullet d & \sim & a\circ c+c\circ d+d\circ a
\end{array}
\qquad
\begin{array}{lcl}
a\bullet b\bullet d & \sim & a\circ b+b\circ d+d\circ a \\
b\bullet c\bullet d & \sim & b\circ c+c\circ d+d\circ b
\end{array}
$$

- mapping the 4 $L3$ elements of FA onto 4 of the $L3$ elements of EA, each trio of faces is associated with the trio of edges which connect the faces:

$$
\begin{array}{lcl}
a\bullet b\bullet c+a\bullet c\bullet d+a\bullet b\bullet d & \sim & b\circ a+c\circ a+d\circ a \\
a\bullet b\bullet c+b\bullet c\bullet d+a\bullet b\bullet d & \sim & a\circ b+c\circ b+d\circ b \\
a\bullet b\bullet c+b\bullet c\bullet d+a\bullet c\bullet d & \sim & a\circ c+b\circ c+d\circ c \\
a\bullet b\bullet d+b\bullet c\bullet d+a\bullet c\bullet d & \sim & a\circ d+b\circ d+c\circ d
\end{array}
$$

- mapping the 6 $L2$ elements of FA onto the 6 $L1$ elements of EA, each pair of faces is associated with the edge that connects the two faces:

$$
\begin{array}{lcl}
a\bullet b\bullet c+a\bullet b\bullet d & \sim & a\circ b \\
a\bullet c\bullet d+a\bullet b\bullet d & \sim & a\circ d \\
b\bullet c\bullet d+a\bullet b\bullet d & \sim & b\circ d
\end{array}
\qquad
\begin{array}{lcl}
a\bullet b\bullet c+a\bullet c\bullet d & \sim & a\circ c \\
a\bullet b\bullet c+b\bullet c\bullet d & \sim & b\circ c \\
b\bullet c\bullet d+a\bullet c\bullet d & \sim & c\circ d
\end{array}
$$

This second example yields the same overall patterns as the first partial mapping: the two special subsets of 4 $L3$ ec's show up when mapping the $L1$ and $L3$ elements of FA, and there is a perfect one-to-one mapping between the 6 fc's of the most complex $L2$ of FA and the 6 ec's on the simplest $L1$ of EA.

8 Conclusion

In this paper we have presented a detailed logical-algebraic and geometrical analysis of the simplest of all polyhedra, viz. the TETRAHEDRON. We introduced the vertex algebra VA for the tetrahedron's zero-dimensional vertex constellations and the face algebra FA for its two-dimensional face constellations, and then discussed two possible isomorphisms between VA and FA. Next, we turned to the more complex edge algebra EA for the tetrahedron's one-dimensional edge constellations, and briefly considered some partial mappings between VA and FA on the one hand and EA on the other. This last step, in particular, definitely deserves much further scrutiny in the future.

Along the way, at least two intruiging issues were raised for further investigation within the framework of Logical Geometry, particularly in relation to the analysis presented in Smessaert & Demey (2014). First of all, is there a natural way of connecting the contrast between negative and positive isomorphisms and the distinction between the opposition and implication geometries? And secondly, what — if any — is the possible connection between the graphical and the logical notions of unconnectedness?

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